

Inhomogeneous Universe in $f(T)$ Theory

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Abstract

We obtain the equations of motions of the $f(T)$ theory considering the Lemaître-Tolman-Bondi's metric for a set of diagonal and non-diagonal tetrads. In the case of diagonal tetrads the equations of motion of the $f(T)$ theory impose a constant torsion or the same equations of the General Relativity, while in the case of non-diagonal set the equations are quite different from that obtained in GR. We show a simple example of an universe dominated by the

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matter for the two cases. The comparison of the mass in the non-diagonal case shows a sort of increased with respect to the diagonal one. We also perform two examples for the non-diagonal case. The first concerns a black hole solution of type Sshwarzschild which presents a temperature higher than that of Schwarzschild, and a black hole in a dust-dominated universe.

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1 Introduction

A possible equivalence between the equations of General Relativity (GR) can be obtained considering a space-time where the curvature contributions vanish and the unique non null contribution is that coming from the antisymmetric part of the connection. This is the scenario of the so-called Weitzenbok's space-time. Through this equivalence, we can analyse the physical phenomena of the Gravitation and the Cosmology, which until now are not clearly known using

the GR. Hence, we can try to easily understand the contribution of the terms of higher order in curvature, added to the Einstein-Hilbert term as can be observed in the common theories, $f(R)$ [2, 3], $f(G)$ [4], $f(R, T)$ [5] and $f(R, G)$ [6]. In the case of the theory equivalent to the GR, the Teleparallel Theory (TT) [1], the attention is now attached to the torsion scalar T which plays an important role in constructing the action of this theory. Hence, as in the case of GR, a generalization of the TT must contain terms of higher order in T that we call $f(T)$ theory [20], where $f(T)$ is an algebraic function of the torsion scalar T .

Several works have been done in this sense these recent months in the $f(T)$ theory, with various interesting results [24, 25]. However, there is not still more progress in introducing new symmetries, as in the case of TT. Therefore, in order to analyse the possible results we propose here to introduce a new symmetry, that of the Lemaître-Tolman's (LT) models. This may help us to understand more about the gravitational and cosmological phenomena cited above.

The solutions called inhomogeneous have various applications in the GR. A good revision of these applications is shown in [7]. A particular case of these solutions is that of LT [8]. The models have been used in redshift drift [10], CMB [11], interpretation of supernova observations [12], averaging [13], formation of black holes [14], of galaxy clusters [15], superclusters [16], cosmic voids [18] and collapse from the perspective of loop quantum gravity [17].

In this paper, we consider the symmetries of the Lemaître-Tolman's metric for comparing the physics of the solution coming from the $f(T)$ theory with the well known results in the GR. To do this, we take two workable choices of tetrads, the diagonal and the non-diagonal one, where we will get the real notion of the main different with respect to the GR.

The paper is organized as follows. In Sec. 2, we explicitly present the equations of motion in $f(T)$ theory. The Sec. 3 is devoted to the characterization of the geometry of an inhomogeneous universe. In Sec. 4, a set of non-diagonal tetrads for the metric of LTB is presented. In Sec 5 we obtain new solutions, where we perform examples of a dust-dominated universe in the subsection 5.1, a Schwarzschild-type black hole solution in the subsection 5.2, a black hole solution in a dust-dominated universe in the subsection 5.3 and other solutions in 5.4. The conclusion and perspective are presented in the Sec. 6.

2 The field equations from $f(T)$ theory

In this section we will develop how obtaining the equations of motions for the $f(T)$ theory and the choice of matter model as an anisotropic fluid.

We start defining the line element as

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} \theta^a \theta^b , \quad (1)$$

$$\theta^a = e^a_\mu dx^\mu , \quad dx^\mu = e_a^\mu \theta^a , \quad (2)$$

where $g_{\mu\nu}$ is the metric of the space-time, η_{ab} is the Minkowski's metric, θ^a are the tetrads and e^a_μ and their inverses e_a^μ are the tetrads matrices that satisfy $e^a_\mu e_a^\nu = \delta_\mu^\nu$ and $e^a_\mu e_b^\mu = \delta_b^a$. The root of the determinant of the metric is given by $\sqrt{-g} = \det[e^a_\mu] = e$. The Weitzenbok's connection is defined by

$$\Gamma_{\mu\nu}^\alpha = e_i^\alpha \partial_\nu e^i_\mu = -e^i_\mu \partial_\nu e_i^\alpha . \quad (3)$$

Through the connection we can define the components of the torsion and the contorsion as

$$T_{\mu\nu}^\alpha = \Gamma_{\nu\mu}^\alpha - \Gamma_{\mu\nu}^\alpha = e_i^\alpha (\partial_\mu e^i_\nu - \partial_\nu e^i_\mu) , \quad (4)$$

$$K^{\mu\nu}_\alpha = -\frac{1}{2} (T^{\mu\nu}_\alpha - T^{\nu\mu}_\alpha - T_\alpha^{\mu\nu}) . \quad (5)$$

For facilitating the description of the Lagrangian and the equations of motion, we can define another tensor from the components of torsion and the contorsion as

$$S_\alpha^{\mu\nu} = \frac{1}{2} \left(K^{\mu\nu}_\alpha + \delta_\alpha^\mu T^{\beta\nu}_\beta - \delta_\alpha^\nu T^{\beta\mu}_\beta \right) . \quad (6)$$

Now, defining the torsion scalar

$$T = T_{\mu\nu}^\alpha S_\alpha^{\mu\nu} , \quad (7)$$

one can define the Lagrangian of the $f(T)$ theory, coupled with the matter as follows

$$\mathcal{L} = e f(T) + \mathcal{L}_{Matter} . \quad (8)$$

The principle of least action leads to the Euler-Lagrange's equations. In order to use these equations we first write the quantities

$$\frac{\partial \mathcal{L}}{\partial e^a_\mu} = f(T) e e_a^\mu + e f_T(T) 4 e_a^\alpha T^\sigma_{\nu\alpha} S_\sigma^{\mu\nu} + \frac{\partial \mathcal{L}_{Matter}}{\partial e^a_\mu} , \quad (9)$$

$$\partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha e^a_\mu)} \right] = -4f_T(T)\partial_\alpha (ee_a^\sigma S_\sigma^{\mu\nu}) - 4ee_a^\sigma S_\sigma^{\mu\alpha} \partial_\alpha T f_{TT}(T) + \partial_\alpha \left[\frac{\partial \mathcal{L}_{Matter}}{\partial(\partial_\alpha e^a_\mu)} \right] , \quad (10)$$

where $f_T(T) = df(T)/dT$ and $f_{TT}(T) = d^2f(T)/dT^2$. The equations of Euler-Lagrange are given by

$$\frac{\partial \mathcal{L}}{\partial e^a_\mu} - \partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha e^a_\mu)} \right] = 0 , \quad (11)$$

which, multiplying by $e^{-1}e^a_\beta/4$, yields

$$S_\beta^{\mu\alpha} \partial_\alpha T f_{TT}(T) + [e^{-1}e^a_\beta \partial_\alpha (ee_a^\sigma S_\sigma^{\mu\alpha}) + T^\sigma_{\nu\beta} S_\sigma^{\mu\nu}] f_T(T) + \frac{1}{4} \delta_\beta^\mu f(T) = 4\pi \mathcal{T}_\beta^\mu , \quad (12)$$

where the energy momentum tensor is given by

$$\mathcal{T}_\beta^\mu = -\frac{e^{-1}e^a_\beta}{16\pi} \left\{ \frac{\partial \mathcal{L}_{Matter}}{\partial e^a_\mu} - \partial_\alpha \left[\frac{\partial \mathcal{L}_{Matter}}{\partial(\partial_\alpha e^a_\mu)} \right] \right\} . \quad (13)$$

For an anisotropic fluid, the energy momentum tensor is given by the expression

$$\mathcal{T}_\beta^\mu = (\rho + p_t) u_\beta u^\mu - p_t \delta_\beta^\mu + (p_r - p_t) v_\beta v^\mu , \quad (14)$$

where u^μ is the four-velocity, v^μ the unit space-like vector in the radial direction, ρ the energy density, p_r the pressure in the direction of v^μ (radial pressure) and p_t the pressure orthogonal to v_μ (tangential pressure). Since we are assuming an anisotropic spherically symmetric matter, one has $p_r \neq p_t$, such that their equality corresponds to an isotropic fluid sphere.

3 The geometry of an inhomogeneous universe

Given the metric of Lemaître-Tolman-Bondi [8, 9]

$$dS^2 = dt^2 - B^2(r, t) dr^2 - A^2(r, t) (d\theta^2 + \sin^2(\theta) d\phi^2) , \quad (15)$$

we can describe this space-time through the following set of diagonal tetrads

$$\{e^a_\mu\} = \text{diag} [1, B(r, t), A(r, t), A(r, t) \sin \theta] , \quad (16)$$

$$\{e_a^\mu\} = \text{diag} [1, B^{-1}(r, t), A^{-1}(r, t), A^{-1}(r, t) \sin^{-1} \theta] , \quad (17)$$

where we define the determinant of the tetrads by $e = \det[e^a_\mu] = A^2 B \sin \theta$. The non null components of the torsion (4) are

$$\begin{cases} T^1_{01} = -T^1_{10} = e_1^1 \partial_0 e^1_1 = B^{-1} \dot{B}, \\ T^2_{12} = -T^2_{21} = e_2^2 \partial_1 e^2_2 = A^{-1} A', \\ T^3_{13} = -T^3_{31} = e_3^3 \partial_1 e^3_3 = A^{-1} A', \\ T^2_{02} = -T^2_{20} = e_2^2 \partial_0 e^2_2 = A^{-1} \dot{A}, \\ T^3_{03} = -T^3_{30} = e_3^3 \partial_0 e^3_3 = A^{-1} \dot{A}, \\ T^3_{23} = -T^3_{32} = e_3^3 \partial_2 e^3_3 = \cot \theta, \end{cases} \quad (18)$$

where the “dot” indicates the derivative with respect to the time t and the “prime” the derivative with respect to the radial coordinate r . The non null components of the contorsion (5) are

$$\begin{cases} K^{01}_1 = -K^{10}_1 = g^{00} T^1_{01} = B^{-1} \dot{B}, \\ K^{02}_2 = -K^{20}_2 = g^{00} T^2_{02} = A^{-1} \dot{A}, \\ K^{03}_3 = -K^{30}_3 = g^{00} T^3_{03} = A^{-1} \dot{A}, \\ K^{12}_2 = -K^{21}_2 = g^{11} T^2_{12} = -\frac{A'}{AB^2}, \\ K^{13}_3 = -K^{31}_3 = g^{11} T^3_{13} = -\frac{A'}{AB^2}, \\ K^{23}_3 = -K^{32}_3 = g^{22} T^3_{23} = -\frac{\cot \theta}{A^2}. \end{cases} \quad (19)$$

We can now calculate the non null components of the tensor $S_\alpha^{\mu\nu}$ in (6), which are given by

$$\begin{cases} S_0^{10} = -S_0^{01} = -\frac{1}{2} g^{11} T^{\beta}_{1\beta} = \frac{A'}{AB^2}, \\ S_0^{20} = -S_0^{02} = -\frac{1}{2} g^{22} T^{\beta}_{2\beta} = \frac{\cot \theta}{2A^2}, \\ S_1^{01} = -S_1^{10} = \frac{1}{2} (K^{01}_1 - T^{\beta 0}_\beta) = -A^{-1} \dot{A}, \\ S_1^{21} = -S_1^{12} = -\frac{1}{2} g^{22} T^{\beta}_{2\beta} = \frac{\cot \theta}{2A^2}, \\ S_2^{02} = -S_2^{20} = \frac{1}{2} (K^{02}_2 - T^{\beta 0}_\beta) = -\frac{1}{2} (A^{-1} \dot{A} + B^{-1} \dot{B}), \\ S_2^{12} = -S_2^{21} = \frac{1}{2} (K^{12}_2 - T^{\beta 1}_\beta) = \frac{A'}{2AB^2}, \\ S_3^{03} = -S_3^{30} = \frac{1}{2} (K^{03}_3 - T^{\beta 0}_\beta) = -\frac{1}{2} (A^{-1} \dot{A} + B^{-1} \dot{B}), \\ S_3^{13} = -S_3^{31} = \frac{1}{2} (K^{13}_3 - T^{\beta 1}_\beta) = \frac{A'}{2AB^2}. \end{cases} \quad (20)$$

Through the definition of the torsion scalar (7) and of the components (18) and (20), one obtains

$$T = 2 \left[\left(\frac{A'}{AB} \right)^2 - 2 \frac{\dot{A}\dot{B}}{AB} - \left(\frac{\dot{A}}{A} \right)^2 \right]. \quad (21)$$

We now obtain two equations that impose constraints to the $f(T)$ theory such that it becomes equivalent to the TT, which is the case where the algebraic function $f(T)$ is a linear function of the torsion scalar T . For the first of them, it is sufficient to put $\beta = 0$ and $\mu = 2$ in (12), which leads to

$$\frac{\cot \theta}{2A^2} \dot{T} f_{TT}(T) = 0 , \quad (22)$$

and for the second, it can be just put $\beta = 1$ and $\mu = 2$ in (12), which yields

$$\frac{\cot \theta}{2A^2} T' f_{TT}(T) = 0 . \quad (23)$$

The equations (49) and (23) inform that, or the torsion scalar is a constant, which does not yield any interesting result for the metric of LTB, or the algebraic function $f(T)$ is linear in T . An imposition of the form (23) has been obtained for a set of diagonal tetrads in the case of a spherically symmetric and static metric [19, 20]. In the next section, we will see the case where it will be taken a set of non-diagonal tetrads for describing the metric of LTB (15).

Substituting the components (18)-(20) and the torsion scalar (21) in (12), for the case of the components $0 - 0$, $1 - 1$ and $2 - 2$, we get the following equations of motion

$$\begin{cases} 8\pi\rho = -2\frac{A''}{AB^2} + 2\frac{A'B'}{AB^3} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{1}{A^2} + \left(\frac{\dot{A}}{A}\right)^2 - \left(\frac{A'}{AB}\right)^2 , \\ -8\pi p_r = 2\frac{\ddot{A}}{A} + \frac{1}{A^2} + \left(\frac{\dot{A}}{A}\right)^2 - \left(\frac{A'}{AB}\right)^2 , \\ -8\pi p_t = -\frac{A''}{AB^2} + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{A'B'}{AB^3} , \end{cases} \quad (24)$$

that are identical to the equations of GR [21]. This is not surprising since the TT theory is dynamically equivalent to the GR [22]. The famous symmetry of the metric of LTB is recuperated when one takes $\beta = 0$ and $\mu = 1$ in (12)

$$\left(\dot{A}'B - A'\dot{B}\right) f_T(T) = 0 , \quad (25)$$

which, after integration leads to

$$B(r, t) = c^{-1}(r) A'(r, t) . \quad (26)$$

The function that appears in (26) as integration constant for the coordinate t can be fixed, as in the case of GR, due to its relationship with the spatial curvature $c(r) = \sqrt{1 - k(r)}$ [21].

A direct application of these results is the so-called limit of the Friedmann-Lemaître-Robertson-Walker's universe. To do this, let us consider the equation of conservation for an energy momentum tensor with null radial and tangential pressures ($p_r = p_t = 0$ in (14)):

$$\nabla_\mu \mathcal{T}^{\mu\nu} = 0 , \quad (27)$$

such that, using (26) and integrating (27), one gets

$$\rho(r, t) = \frac{c(r)\rho_0(r)}{A^2(r, t)A'(r, t)} , \quad (28)$$

where $\rho_0(r)$ is an algebraic function of r , coming from the integration in t . The first equation in (24) can be rewritten as

$$8\pi\rho A^2 A' = \left[A \left(1 - c^2(r) + \dot{A}^2 \right) \right]' . \quad (29)$$

Defining the mass of the diagonal case as

$$M_D(r) = 8\pi \int_0^r c(y)\rho(y, t)A^2(y, t)B(y, t)dy = 8\pi \int_0^r c(y)\rho_0(r)dy , \quad (30)$$

the equation (29) can be integrated, yielding

$$\dot{A}^2 = \frac{M_D(r)}{A} + c^2(r) - 1 . \quad (31)$$

A particular case of this equation is when $c(r) = 1$, where the integration leads to

$$A(r, t) = \left[d(r) + \frac{3}{2}\sqrt{M_D(r)}t \right]^{2/3} , \quad (32)$$

where $d(r)$ is an algebraic function of r . Putting $d(r) = r^{3/2}$ and $M_D(r) = 8\pi\bar{\rho}_0 r^3/3$, where $\bar{\rho}_0$ is a constant, we re-obtain the equations of Friedmann for an universe dominated by the matter [23], where $A(r, t) = ra(t)$, $a(t) = [1 + \sqrt{6\pi\bar{\rho}_0}t]^{2/3}$, $\rho(t) = \bar{\rho}_0/a^3(t)$ and $t_B = -(6\pi\bar{\rho}_0)^{-1/2}$ representing the Big Bang.

In the next section we will perform the calculus about the equations of motion for a set of non-diagonal tetrads.

4 A set of non-diagonal tetrad

We can also project in the tangent space to the LTB's metric (15) through a set of non-diagonal tetrads as follows

$$\{e^a_{\mu}\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & B(r, t) \sin \theta \cos \phi & A(r, t) \cos \theta \cos \phi & -A(r, t) \sin \theta \sin \phi \\ 0 & B(r, t) \sin \theta \sin \phi & A(r, t) \cos \theta \sin \phi & A(r, t) \sin \theta \cos \phi \\ 0 & B(r, t) \cos \theta & -A(r, t) \sin \theta & 0 \end{bmatrix} , \quad (33)$$

whose inverse is

$$\{e_a^\mu\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & B^{-1}(r, t) \sin \theta \cos \phi & B^{-1}(r, t) \sin \theta \sin \phi & B^{-1}(r, t) \cos \theta \\ 0 & A^{-1}(r, t) \cos \theta \sin \phi & A^{-1}(r, t) \cos \theta \cos \phi & -A^{-1}(r, t) \sin \theta \\ 0 & -A^{-1}(r, t) \sin^{-1} \theta \sin \phi & A^{-1}(r, t) \sin^{-1} \theta \cos \phi & 0 \end{bmatrix}. \quad (34)$$

By using (33) and (34), the non null components of the torsion (4) are calculated as:

$$\left\{ T^1_{01} = B^{-1} \dot{B}, T^2_{02} = T^3_{03} = A^{-1} \dot{A}, T^2_{12} = T^3_{13} = -\frac{B-A'}{A} \right\}. \quad (35)$$

The non null components of the contorsion are

$$\left\{ K^{01}_1 = B^{-1} \dot{B}, K^{02}_2 = K^{03}_3 = A^{-1} \dot{A}, K^{12}_2 = K^{13}_3 = \frac{B-A'}{AB^2} \right\}. \quad (36)$$

The non null components of $S_\alpha^{\mu\nu}$ are

$$\left\{ S_0^{01} = 2S_2^{21} = 2S_3^{31} = \frac{B-A'}{AB^2}, S_1^{10} = A^{-1} \dot{A}, S_2^{20} = S_3^{30} = \frac{1}{2}(A^{-1} \dot{A} + B^{-1} \dot{B}) \right\}. \quad (37)$$

Taking into account the components (35) and (37), the torsion scalar (7) becomes

$$T = -2 \left[2 \frac{\dot{A}\dot{B}}{AB} + \left(\frac{\dot{A}}{A} \right)^2 - \frac{1}{A^2} + \frac{2A'}{A^2 B} - \left(\frac{A'}{AB} \right)^2 \right]. \quad (38)$$

Through (33)-(38), the equations of motion (12) are given by

$$\begin{cases} \frac{(B-A')}{AB^2} T' f_{TT} + f_T \left[\frac{T}{2} + \frac{1}{A^2} + 4 \frac{\dot{A}\dot{B}}{AB} + \frac{A'B'}{AB^3} - \frac{A'}{A^2 B} - \frac{A''}{AB^2} \right] + \frac{f}{4} = 4\pi\rho, \\ -\frac{(B-A')}{AB^2} \dot{T} f_{TT} + \frac{f_T}{AB} (A'\dot{B} - \dot{A}'B) = 0, \\ -\frac{\dot{A}}{A} T' f_{TT} - \frac{f_T}{AB^3} (A'\dot{B} - \dot{A}'B) = 0, \\ \frac{\dot{A}}{A} T' f_{TT} + f_T \left[\frac{T}{2} + \frac{\ddot{A}}{A} + 3 \frac{\dot{A}\dot{B}}{AB} + \frac{1}{A^2} - \frac{A'}{A^2 B} \right] + \frac{f}{4} = -4\pi p_r, \\ \frac{1}{2} \left[\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) \dot{T} + \frac{B-A'}{AB^2} T' \right] f_{TT} + \frac{f_T}{2} \left[\frac{T}{2} + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + 5 \frac{\dot{A}\dot{B}}{AB} + \frac{A'B'}{AB^3} - \frac{A''}{AB^2} \right] + \frac{f}{4} = -4\pi p_t. \end{cases} \quad (39)$$

The equations of motion of the non-diagonal case are quite different from the previous ones of the diagonal case. This is the proof of the dependence on the frame in $f(T)$ theory [26]. In such a situation, the physics that results from this set of equations must present new data that may be important in understanding the subjects that still require attention in Cosmology and Astrophysics.

5 New solutions

5.1 Dust-dominated universe

Let us make a simplified example of the analysis of these equations. Taking the linear case $f(T) = T$, the second and third equations of (39) yield again the constraint (26) well known in the metric of LTB. In order to compare with the example taken in the diagonal case, we put $p_r = p_t = 0$ in (14) and rewrite the first equation of (39) as

$$8\pi\rho A^2 A' = 8\pi\rho_D A^2 A' + 2A' \left[2c(r) - 1 - c^2(r) + \dot{A}^2 \right] , \quad (40)$$

where ρ_D is the energy density of the diagonal case in (29). Using the equation of conservation (27), we re-obtain (28), that suggests the same definition (30) for the mass in the non-diagonal case. But here we have the following identity:

$$\begin{aligned} M(r, t) &= 8\pi \int_0^r c(y) \rho(y, t) A^2(y, t) B(y, t) dy = M_D(r) \\ &\quad + 2 \int_0^r A'(y, t) \left[2c(y) - 1 - c^2(y) + \dot{A}^2(y, t) \right] dy . \end{aligned} \quad (41)$$

Here the mass of the non-diagonal can depend on the time in general. Comparing with the diagonal case, it appears that the mass of the non-diagonal case possesses an increased (decreased) due to the non diagonal description of the matrix of the tetrads in (33).

Now, let us look at for the particular case $c(r) = 1$ and, as in the diagonal case, one supposes $A(r, t) = ra(t)$, with $a(t) = [1 + \sqrt{6\pi\bar{\rho}_0}t]^{2/3}$. Since $M_D(r) = A\dot{A}^2 = 8\pi\bar{\rho}_0 r^3/3$, from (41), one gets the following identity

$$M(r) = M_D(r) + \frac{2}{3} M_D(r) . \quad (42)$$

This shows us that a significant value (2/3) of the mass (or energy density) of the diagonal case is increased as contribution in the non-diagonal case. Note that the addition comes from the contribution of the set of the off-diagonal terms of the non-diagonal tetrads matrix.

5.2 Black hole solution

For an exterior solution, in the vacuum, we get $\rho = 0$ in the first equation of (39), which for $c(r) = 1$ yields

$$\left[A\dot{A}^2 \right]' + 2A'\dot{A}^2 = 0, \quad (43)$$

which can be rewritten as

$$3 \left[A \dot{A}^2 \right]' - 2A \left[\dot{A}^2 \right]' = 0 . \quad (44)$$

A solution of (44) is given by

$$A(r, t) = [k_1(t) + k_2(r)]^{2/5} . \quad (45)$$

Making the coordinate transformation $x(r, t) = A(r, t)$, the line element (15) becomes

$$dS^2 = \left(1 - \frac{4\dot{k}_1^2}{25x^3} \right) dt^2 + \frac{4\dot{k}_1}{5x^{3/2}} dt dx - dx^2 - x^2 d\Omega^2 , \quad (46)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. Carrying out a new change of coordinates

$$dT(x, t) = b_1(x, t) dt + b_2(x, t) dx , \quad (47)$$

the line element (46) turns into

$$\begin{aligned} dS^2 = & \frac{1}{b_1^2} \left(1 - \frac{4\dot{k}_1^2}{25x^3} \right) dT^2 - \left[\frac{2b_2}{b_1^2} \left(1 - \frac{4\dot{k}_1^2}{25x^3} \right) - \frac{4\dot{k}_1}{5x^{3/2}b_1} \right] dT dx + \\ & - \left[1 + \frac{4\dot{k}_1 b_2}{5x^{3/2}b_1} - \left(\frac{b_2}{b_1} \right)^2 \left(1 - \frac{4\dot{k}_1^2}{25x^3} \right) \right] dx^2 - x^2 d\Omega^2 . \end{aligned} \quad (48)$$

Imposing

$$g_{TT} \equiv 1 - \frac{2M}{x} , \quad g_{Tx} \equiv 0 , \quad k_1(t) \equiv 5\sqrt{2M^3}t , \quad (49)$$

the line element (48) reads

$$dS^2 = \left(1 - \frac{2M}{x} \right) dT^2 - \left[1 - \left(\frac{2M}{x} \right)^3 \right]^{-1} dx^2 - x^2 d\Omega^2 , \quad (50)$$

where

$$b_1(r) = \sqrt{\frac{x^3 - 8M^3}{x^2(x - 2M)}} , \quad b_2(r) = 2\sqrt{\frac{2M^3 x}{(x - 2M)(x^3 - 8M^3)}} . \quad (51)$$

The expression (50) is a black hole solution of type-Schwarzschild, with the mass M and horizon in $x_H = 2M$, but the Hawking temperature defined as $T_H = \sqrt{3}/8\pi M > T_{Schwarzschild}$. The usual case of Lemaitre-Tolman-Bondi, the solution is exactly that of Schwarzschild. Then, if we consider the increase of the mass as seen in the previous subsection, $M = 5M_D/3$, we get the inequality $T_H = (3\sqrt{3}/5) T_{HD} > T_{HD} = (1/8\pi M_D)$, where T_{HD} and M_D are the temperature and the mass of the diagonal case.

5.3 Black hole in a dust-dominated universe

We can consider the results of the two previous subsections and generalize a solution for the black hole immersed in a dust-dominated universe. We proceed as follows. We need a solution such that when the mass M is identically null, the solution corresponding to a universe dominated by the dust is recovered, while when we put $\bar{\rho}_0 = 0$, the solution characterizes a black hole of type-Schwarzschild. Thus, just consider the simple linear combination

$$A(r, t) = rM \left(1 + 5\sqrt{2M^3 t} r^{-5/2}\right)^{2/5} + \bar{\rho}_0 r \left(1 + \sqrt{6\pi\bar{\rho}_0 t}\right)^{2/3}, \quad (52)$$

which represents a black hole in a dust-dominated universe. The energy density can be easily calculated by the equation (40), but we do not present this step due to its too long form. However, the limit $r, t \rightarrow 0$, leads to an infinite energy density, as at the Big Bang, and $t \rightarrow t_0$ leads to the current energy density, $\tilde{\rho}_0$. From the equation (40), isolating ρ , we see that the energy density is always positive, for $t \geq -\sqrt{6\pi\bar{\rho}_0}$ and $r \geq 0$.

5.4 Other exact solutions

Let us return to the system (39). This system contents 5 equations for 6 unknown functions. To solve it, we need one more additional equation. In any case, the system (39) has the very complicated form. So that the finding its solutions is very hard job. For that reason let us simplify the task considering some particular cases.

5.4.1 Static solution

Here we now assume that $A = A(r)$ and $B = B(r)$ that corresponds to the static case. Then the system (39) takes the form

$$\begin{cases} \frac{(B-A')}{AB^2} T' f_{TT} + f_T \left[\frac{T}{2} + \frac{1}{A^2} + \frac{A'B'}{AB^3} - \frac{A'}{A^2 B} - \frac{A''}{AB^2} \right] + \frac{f}{4} = 4\pi\rho, \\ f_T \left[\frac{T}{2} + \frac{1}{A^2} - \frac{A'}{A^2 B} \right] + \frac{f}{4} = -4\pi p_r, \\ \frac{B-A'}{2AB^2} T' f_{TT} + \frac{f_T}{2} \left[\frac{A'B'}{AB^3} - \frac{A''}{AB^2} \right] + \frac{f}{4} = -4\pi p_t. \end{cases} \quad (53)$$

Note that in this case

$$T = 2 \left[\frac{1}{A^2} - \frac{2A'}{A^2 B} + \left(\frac{A'}{AB} \right)^2 \right]. \quad (54)$$

From (53) we get

$$\begin{cases} f_T \left[\frac{T}{2} + \frac{1}{A^2} - \frac{A'}{A^2 B} \right] - \frac{f}{4} = 4\pi(\rho + 2p_t), \\ f_T \left[\frac{T}{2} + \frac{1}{A^2} - \frac{A'}{A^2 B} \right] + \frac{f}{4} = -4\pi p_r, \\ \frac{B-A'}{2AB^2} T' f_{TT} + \frac{f_T}{2} \left[\frac{A'B'}{AB^3} - \frac{A''}{AB^2} \right] + \frac{f}{4} = -4\pi p_t. \end{cases} \quad (55)$$

From the first two equations of this system (55) we obtain

$$f = -8\pi(\rho + 2p_t + p_r). \quad (56)$$

Finally we note that the system (55) contents 3 equations for 6 unknown functions $(A, B, f, \rho, p_t, p_r)$. So to solve this system we need 3 additional equations. In the particular case of the vacuum, for $f(T) = \sum_{n=1}^N a_n T^n$, with $a_0 = 0$ (without cosmological constant), one gets $f(T) = 0$, implying that $T = 0$. By integrating (54) for null torsion, we obtain $A'(r) = B(r)$. Making a change of coordinates $R = A(r)$, the Minkowski's space metric is recovered in (15). This result is consistent with the consideration of the equations in vacuum, as shown in [28].

5.4.2 Time dependent solution

Now we try to get a time dependent solution of the system (39). From the second and third equations of (39) we get

$$f_T = D_1 \exp \left[\int dT \frac{(A'\dot{B} - \dot{A}'B)B}{(B - A')\dot{T}} \right] \quad (57)$$

and

$$f_T = D_2 \exp \left[- \int dT \frac{A'\dot{B} - \dot{A}'B}{B^3 \dot{A}T'} \right], \quad (58)$$

respectively. Here D_i ($i = 1, 2$) are integration constants. Hence we come to the following constraint for the metric that is for the functions A, B :

$$\int dT \frac{(A'\dot{B} - \dot{A}'B)B}{(B - A')\dot{T}} = D_3 - \int dT \frac{A'\dot{B} - \dot{A}'B}{B^3 \dot{A}T'}, \quad (59)$$

where $D_3 = \ln \frac{D_2}{D_1}$. Note that if $D_1 = D_2$, this constraint takes the form

$$B^4 \dot{A}T' = -(B - A')\dot{T}. \quad (60)$$

Note that Eqs. (57)-(58) tell us that the function f in terms of the metric A, B expressed as

$$f = D_3 + D_1 \int dT \exp \left[\int dT \frac{(A'\dot{B} - \dot{A}'B)B}{(B - A')\dot{T}} \right] \quad (61)$$

or

$$f = D_4 + D_2 \int dT \exp \left[- \int dT \frac{A' \dot{B} - \dot{A}' B}{B^3 \dot{A}' T'} \right], \quad (62)$$

respectively. Here D_3 and D_4 are integration constants. Now our aim is to express ρ, p_r, p_t in terms of the metric functions. To do it, we rewrite the system (39) as

$$\begin{cases} K_2 f_{TT} + K_1 f_T + \frac{f}{4} = 4\pi\rho, \\ -\frac{(B-A')}{AB^2} \dot{T} f_{TT} + \frac{f_T}{AB} (A' \dot{B} - \dot{A}' B) = 0, \\ -\frac{\dot{A}}{A} T' f_{TT} - \frac{f_T}{AB^3} (A' \dot{B} - \dot{A}' B) = 0, \\ N_2 f_{TT} + N_1 f_T + \frac{f}{4} = -4\pi p_r, \\ M_2 f_{TT} + M_1 f_T + \frac{f}{4} = -4\pi p_t. \end{cases} \quad (63)$$

where

$$\begin{cases} K_2 = \frac{(B-A')}{AB^2} T', & K_1 = \left[\frac{T}{2} + \frac{1}{A^2} + 4 \frac{\dot{A}\dot{B}}{AB} + \frac{A'B'}{AB^3} - \frac{A'}{A^2 B} - \frac{A''}{AB^2} \right], \\ N_2 = \frac{\dot{A}}{A} T', & N_1 = \left[\frac{T}{2} + \frac{\ddot{A}}{A} + 3 \frac{\dot{A}\dot{B}}{AB} + \frac{1}{A^2} - \frac{A'}{A^2 B} \right], \\ M_2 = \frac{1}{2} \left[\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) \dot{T} + \frac{B-A'}{AB^2} T' \right], & M_1 = \frac{1}{2} \left[\frac{T}{2} + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + 5 \frac{\dot{A}\dot{B}}{AB} + \frac{A'B'}{AB^3} - \frac{A''}{AB^2} \right]. \end{cases} \quad (64)$$

Let us eliminate f_{TT} from (63). To do it, we can use the second or third equations of the system (63). As result we come to the equations

$$f_{TT} = \frac{(A' \dot{B} - \dot{A}' B) B}{(B - A') \dot{T}} f_T \equiv L_1 f_T \quad (65)$$

and

$$f_{TT} = -\frac{A' \dot{B} - \dot{A}' B}{B^3 \dot{A}' T'} f_T \equiv L_2 f_T. \quad (66)$$

Note that from these two equations follows that $L_1 = L_2$ that is equivalent to the constraint (60). Using the equations (65) or (66), from (63) we get

$$\begin{cases} (K_2 L_1 + K_1) f_T + \frac{f}{4} = 4\pi\rho, \\ (N_2 L_1 + N_1) f_T + \frac{f}{4} = -4\pi p_r, \\ (M_2 L_1 + M_1) f_T + \frac{f}{4} = -4\pi p_t. \end{cases} \quad (67)$$

Hence finding for example f_T as

$$f_T = \frac{4\pi\rho - f/4}{K_2 L_1 + K_1}, \quad (68)$$

we finally come to the following formula for $f(T)$:

$$f = -16\pi \left(1 - \frac{N_2 L_1 + N_1}{K_2 L_1 + K_1} \right)^{-1} \left(\frac{N_2 L_1 + N_1}{K_2 L_1 + K_1} \rho + p_r \right) \quad (69)$$

or

$$f = 16\pi (3 - W)^{-1} [(1 - W) \rho - p_r - p_t], \quad (70)$$

where

$$W = \frac{(K_2 + N_2 + M_2)L_1 + K_1 + N_1 + M_1}{K_2L_1 + K_1}. \quad (71)$$

Once again we have here an algebraic function $f(T)$ which depends on the matter content in (70). This confirms again that a possibility of getting a consistency of this theory is considering a matter content depending on the algebraic function $f(T)$ and its derivatives, as shown in [20].

6 Conclusion

We obtained the equations of motion for the $f(T)$ theory in (12). We first took a set of diagonal tetrads for the case of the Lemaître-Tolman-Bondi's (LTB) metric and obtained the same results as that of the General Relativity (GR) in (24). This does not seem surprising since it is well known that the Teleparallel Theory is dynamically equivalent to the GR [1], which is particularly our case here. We explored a particular case of an universe dominated by the matter for comparing it with the case of a set of non-diagonal tetrads.

Afterwards, we chose a new set of non-diagonal tetrads for projecting the metric of LTB in the tangent space and obtained new equations of motion of this case. This result, that the $f(T)$ theory possesses a dependence on the frame in its description [26], also is not surprising, and the fact that the equations in the non-diagonal frame being different from that of the diagonal one was already expected. We explained the same example of an universe dominated by the matter and we noted that the increased (decreased) in the mass (or energy density) is possibly dependent on the time, what is drastically different from the GR. We also perform the example of a black hole solution, which is of type-Schwarzschild and a slightly higher Hawking temperature. Our last example is that of black hole in a dust-dominated universe, which produces the same result as in the case of GR.

Through a set of non-diagonal tetrads we still were able to make various analysis already made in the GR, as the evolution of the black holes apparent horizon (AH) and cosmic AH [23], CMB [11, 27] and many other possibilities, but which will be minutely addressed in a future

work. Hence, we make possible the analysis of the other usual cosmological and astrophysical phenomena, already realized in the GR, but which still have some obscure points to be explained.

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